Adjustment of complex probabilistic models and estimation of confidence intervals in a discrete manner

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Abstract: This article shows that complex probabilistic models can be properly adjusted from feedback data, if an efficient optimization method is used to overcome the multiple optima of the likelihood function.
It proposes a generic method to estimate confidence intervals by inverting the Fisher matrix calculated in a discrete manner in order to avoid the derivation of analytical expressions which are specific to each model used.

Keywords: Adjustment, global optimization, genetic algorithms, confidence interval, Fisher matrix, discrete calculation

1. INTRODUCTION

Developed by Ronald Aylmer Fisher, the maximum likelihood estimation is a common statistical method used to estimate the parameters of a probabilistic model from a given sample. It consists in choosing parameter values that maximize the likelihood of the observations, i.e., the probability of observing them. In the case of uncensored data, the expression of the likelihood is:

$$L(X,\theta) = \prod_{i=1}^{n} f(xi,\theta) \tag{1}$$

With $f(xi,\theta)$ = Probability density of the model if X is a continuous variable.

It is equivalent and simpler to maximize the log-likelihood for which the product becomes a sum.

This optimization problem is usually solved by cancelling the first derivative (if the likelihood function is differentiable), verifying that the second derivative is negative. It also deals with local optimization techniques such as pseudo gradient or non-linear simplex (Nelder Mead algorithm).

But the adjustment can not be achieved by a simple optimization technique when there are many local optima. That's why most of the existing tools are limited to the usual laws of probability and sometimes give erroneous results when the models become more complex.

Stochastic methods like heuristic or Meta heuristics have largely proved their effectiveness in finding global optima, although the optimality of the solutions can not be guaranteed or demonstrated. Also, it appears interesting to use them to make adjustments, especially because they require no a priori knowledge on the functions to be optimized, except for the result obtained for each configuration of parameters.

Therefore, an adjustment function for complex probabilistic models was developed from a global optimization tool (GENCAB of CAB INNOVATION) based on genetic algorithms, differential evolution and non-linear simplex [2]. This hybridization of global and local techniques allows the convergence to be accelerated and the tool to become robust to the variety of problems.

The purpose of this communication is to show that complex probabilistic models can be properly adjusted from feedback data, if an efficient optimization method is used to overcome the multiple optima of the likelihood function.

This article also proposes a generic method to estimate confidence intervals, for parameters or functions of these parameters (quantile, distribution function ...), by inverting the Fisher matrix. The latter is calculated in a discrete manner in order to avoid the derivation of analytical expressions which are specific to each model used.

2. ADJUSTMENT OF COMPLEX PROBABILISTIC MODELS

Using a global optimization tool allows complex models – that were difficult or impossible to treat – to be adjusted as illustrated by the examples below.
This ability to fit complex models can be easily demonstrated by using a set of simulated data from a model with a known configuration of parameters and by obtaining the same parameters configuration when fitting this simulated data set with the tool.

2.1. Reliability models

In a feedback analysis, it is not always easy to separate the early failure, occasional failures and breakdowns of wear. Thus we can adjust them globally using a Multi-phases reliability model.

Bertholon model (Exponential + Weibull)

$\lambda(t) = \lambda + SI(t>T;\beta*(t-T)^{\beta-1}/\eta^{\beta};0)$

$R(t) = exp(-\lambda t)*exp(-[max(0, t-T)/\eta]^{\beta})$

$f(t) = \lambda(t)*R(t)$

Simulation : MIN(-LN(ALEA())/ λ;T+ η(-LN(ALEA()))^(1/ β))*

Adjustment from 100 simulated values

	Initial	Ajusted
λ :	1,00E-05	1,13E-05
β :	2	1,81251122
η :	2000	1861,10116
T :	10000	10250,7697

LN Likelihood : -864,9810628

Simulated values	Lambda	R(t)	LN(f(t))
10261			
10714	0,000326002	0,817257264	-8,23040905
11703	0,000807587	0,462692598	-7,892151769
10766	0,000354309	0,802976014	-8,164771803
12263	0,001049005	0,275015539	-8,15084067
1650	1,1327E-05	0,981481306	-11,40701056
13442	0,001520746	0,060205827	-9,298540254
10687	0,000311123	0,824268538	-8,268580526
12471	0,00113547	0,219046502	-8,299180248
12088	0,000975239	0,328174176	-8,047038707
11168	0,000559369	0,667761413	-7,892525243
10431	0,000157638	0,87570204	-8,887941409
11778	0,000840772	0,435007351	-7,91358232
12725	0,001238937	0,161985564	-8,513749455
10756	0,000349038	0,805727178	-8,176338887
11116	0,000533929	0,687059569	-7,910581423
11739	0,000823651	0,449249291	-7,901940527
7934	1,1327E-05	0,914051809	-11,47818627
11994	0,000934827	0,359121934	-7,999241941
1014	1,1327E-05	0,988576443	-11,39980755

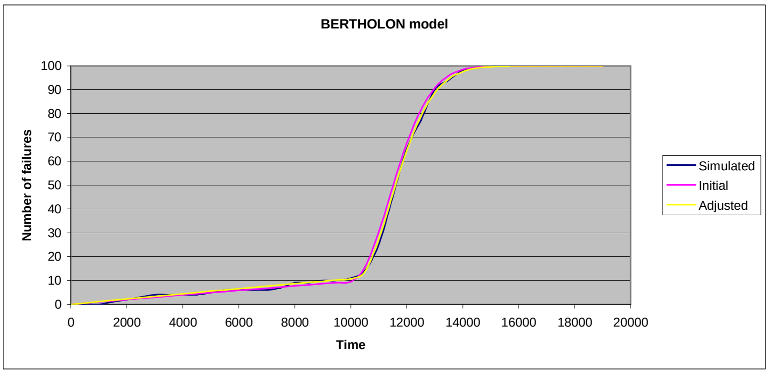

Figure 1. Bertholon model fit from simulated data

Aging model with 3 phases (Weibull + Exponential + Weibull)

$\lambda(t) = SI(t<T_1 ; \beta_1*t^{\beta 1-1}/\eta_1^{\beta 1} ; 0) + \lambda + SI(t>\gamma_2 ; \beta_2*(t-\gamma_2)^{\beta 2-1}/\eta_2^{\beta 2} ; 0)$

$R(t) = exp(-[min(t ;T_1)/\eta_1]^{\beta 1}) * exp(-\lambda t) * exp(-[max(0 ; t-\gamma_2)/\eta_2]^{\beta 2})$

$f(t) = \lambda(t)*R(t)$

Simulation :

*Ti = η₁ *(-LN(ALEA()))^(1/ β1)*

T = si(Ti<T1;Ti;MIN(-LN(ALEA())/ λ; γ2+ η2(-LN(ALEA()))^(1/ β2)))*

Adjustment from 100 simulated values

	Initial	Ajusted
β 1 :	0,6	6,08E-01
η 1 :	10000	8641,566657
T 1 :	1000	954,4707732
λ :	1,00E-05	1,39405E-05
β 2 :	2	1,75281096
η 2 :	2000	1539,914646
γ 2 :	15000	15355,3812

LN Likelihood : -883,6163295

Ti	Simulated values	λ(t)	R(t)	LN(f(t))
15532	16181			
25552,7	16631	0,001001762	0,297376305	-8,118751463
495,393	495	0,000229754	0,832957234	-8,561273633
12795,6	17390	0,001417583	0,11842926	-8,692241517
23186	15614	0,000311293	0,592381589	-8,598378681
21871,7	16457	0,000898704	0,350755633	-8,062222014
7016,9	17243	0,001340984	0,144876339	-8,546226444
10321,6	17224	0,00133058	0,148727123	-8,527782678
3393,09	16238	0,000762273	0,421053427	-8,044201209
187,877	188	0,000329568	0,904688817	-8,117890814
2732,6	16389	0,000857236	0,37235934	-8,049693404
13,8624	14	0,000890997	0,980009642	-7,043361845
2801,1	18797	0,002099266	0,009866492	-10,78477854
9270,69	5862	1,39405E-05	0,70911919	-11,52444637
7434,45	16461	0,000901151	0,349479715	-8,06314764
2674,32	17546	0,00149815	0,094261722	-8,865204393
45815,4	16463	0,000902243	0,348910219	-8,063567441
1524,84	16088	0,000664917	0,468345928	-8,074396046
8044,55	15676	0,000363088	0,580205456	-8,465238471
369,338	369	0,000256088	0,858761074	-8,422254791
20759,7	16489	0,000917766	0,340817113	-8,069977117

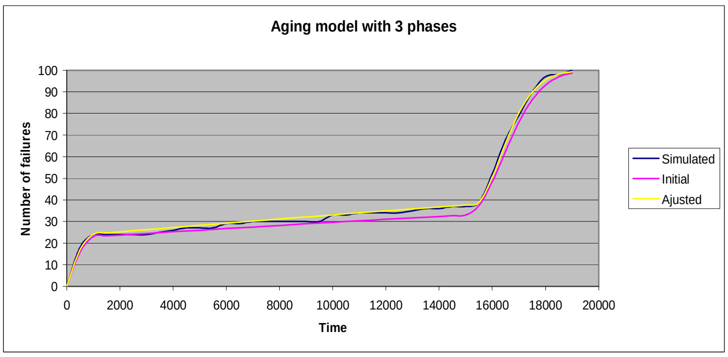

Figure 2. Fit of the full bathtub curve from simulated data

The Bertholon [1] model combines an exponential law and a Weibull law to imitate the second and third part of the bathtub curve (occasional failures and wear). Figure 1 shows the good fit of such a model computed from 100 previously simulated values. This returns approximately the same values than the four initial model parameters used for simulation.

By combining an exponential and 2 Weibull laws, this model can be generalized in a model with seven parameters characterizing the three phases of the bathtub curve as shown in Figure 2.

The expressions of the probability densities used to calculate the log likelihood functions and the functions used to simulate data are given in the figures above, in the same way as the curves representing the distribution functions of the theoretical model, the experimental model from simulated data and the fitted model.

Similarly, it is often difficult to separate data for multiple concurrent failures phenomena. Thus we can adjust them globally using a Multi-modes reliability model as multi-Weibull laws. Figure 3 shows the fit of a bi-Weibull model computed from simulated values.

Combination of two phenomena of wear

	Initial		Ajusted
Bêta 1 :	1,2	Bêta1:	1,15662
Sigma 1 :	2000	Sigma1:	1940,99
Gamma 1 :	100	Gamma1:	122,627
Bêta 2 :	3,5	Bêta2:	3,63118
Sigma 2 :	500	Sigma2:	506,5
Gamma 2 :	1000	Gamma2:	995,441

$R(t) = R_1(t) * R_2(t) = \exp(-[\max(0 ; t-\gamma_1)/\eta_1]^{\beta 1} * \exp(-[\max(0 ; t-\gamma_2)/\eta_2]^{\beta 2})$

$f(t) = \lambda(t) * R(t)$ with $\lambda(t) = \lambda_1(t) + \lambda_2(t) = \beta_1 \max(0 ; t-\gamma_1)^{\beta i-1} / \sigma_i^{\beta 1} + \beta_1 \max(0 ; t-\gamma_1)^{\beta i-1} / \sigma_i^{\beta 1}$

$Ti = = MIN(\gamma_1 + \sigma_1*(-LN(ALEA())^\wedge(1/\beta_1));\gamma_2 + \sigma_2*(-LN(ALEA())^\wedge(1/\beta_2)))$

LN Likelihood: -1807,8

T simulated	T	Lbd1	Lbd2	R1	R2	f(T)		LN(f(T))
1397	1134	0,00054	0,00024	0,62471	0,99101	0,00048	0,00048	-7,64259
	997	0,00053	3,3E-09	0,67179	1	0,00035	0,00035	-7,94807
	1550	0,00057	0,00909	0,49627	0,24989	0,0012	0,0012	-6,7278
	1220	0,00054	0,00084	0,59639	0,94947	0,00078	0,00078	-7,15095
	801	0,00051	0	0,74354	1	0,00038	0,00038	-7,88648
	1551	0,00057	0,00914	0,49589	0,24683	0,00119	0,00119	-6,73485
	1507	0,00057	0,00735	0,5085	0,35542	0,00143	0,00143	-6,55006
	1550	0,00057	0,00909	0,49626	0,24978	0,0012	0,0012	-6,72805
	1428	0,00056	0,00474	0,53141	0,56801	0,0016	0,0016	-6,43714
	1388	0,00056	0,00368	0,54343	0,6719	0,00155	0,00155	-6,47251
	865	0,00051	0	0,71961	1	0,00037	0,00037	-7,90501
	775	0,0005	0	0,75318	1	0,00038	0,00038	-7,87962
	1362	0,00056	0,00307	0,55131	0,73308	0,00147	0,00147	-6,5252
	1086	0,00053	7,6E-05	0,64113	0,99811	0,00039	0,00039	-7,84804
	1624	0,00057	0,01263	0,47581	0,11258	0,00071	0,00071	-7,25439
	1338	0,00055	0,00256	0,5589	0,78568	0,00137	0,00137	-6,59549
	1407	0,00056	0,00415	0,53778	0,62443	0,00158	0,00158	-6,44872
	1180	0,00054	0,0005	0,60935	0,97471	0,00062	0,00062	-7,38417
	759	0,0005	0	0,75919	1	0,00038	0,00038	-7,87552
	174	0,00034	0	0,98515	1	0,00033	0,00033	-8,00943
	1394	0,00056	0,00381	0,54184	0,65884	0,00156	0,00156	-6,4646
	1403	0,00056	0,00405	0,53902	0,6351	0,00158	0,00158	-6,4529
	1396	0,00056	0,00386	0,54123	0,65375	0,00156	0,00156	-6,4618

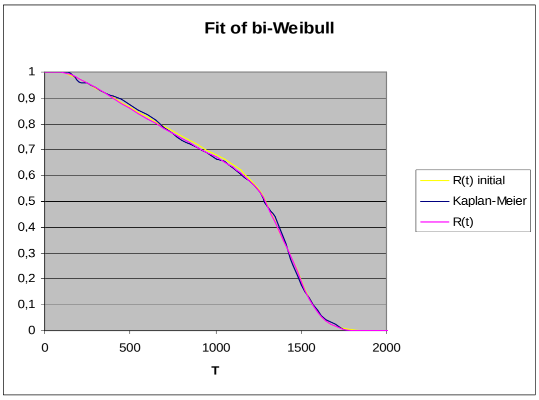

Figure 3. Fit of bi-Weibull from simulated data

2.2. Acceleration Models

Models of accelerated aging are used to reduce the duration of reliability tests by increasing the level of stress. But they can also be used to adjust models of reliability from data obtained under various operational conditions, as shown by the following examples.

Adjustment *Maximum Likelihood*

Acceleration: ARRHENIUS
Ea : 0,565791861

Probability law: WEIBULL (2 parameters)
Bêta : 1,85254944
Sigma : 4228,464162

$$FA = e^{\frac{Ea}{k}\left(\frac{1}{T1} - \frac{1}{T2}\right)}$$

LN Likelihood: -76,177872

Uncensored

LN K (uncensored): -76,177872

Variable	Temperature °K (Ref : 298)	Acceleration factor	AF * ti	Rate : λ(ti)	R(ti) = 1-F(ti)	Densité : f(ti)	Ln(f(ti))
5264,400662	298	1	5264,400662	0,000528106	0,222969125	0,000117751	-9,04693548
7918,411204	298	1	7918,411204	0,000747944	0,040886343	3,05807E-05	-10,3951422
2426,0202	298	1	2426,0202	0,00027282	0,699580676	0,00019086	-8,56397065
1987,125182	298	1	1987,125182	0,000230137	0,781253646	0,000179796	-8,62368938
154,8324887	323	5,503490842	852,1191836	0,000615349	0,949870409	0,000584502	-7,44474991
650,2377125	323	5,503490842	3578,577296	0,00209141	0,479948285	0,001003769	-6,90399362
219,2371129	323	5,503490842	1206,569443	0,000827754	0,906685864	0,000750513	-7,19475418
62,8377678	373	83,95971291	5275,840944	0,044421765	0,221624837	0,009844966	-4,62079498
48,20016738	373	83,95971291	4046,872215	0,035432827	0,397761955	0,014093831	-4,26201812
29,63780026	373	83,95971291	2488,381201	0,023406968	0,687651091	0,016095827	-4,12919521
73,44463393	373	83,95971291	6166,39038	0,050739587	0,133777213	0,006787801	-4,99262832

Figure 4. Fit of Weibull distribution coupled with the Arrhenius model (temperature)

Thus, it is possible to adjust various probability distributions (exponential, Weibull, lognormal ...) associated with different acceleration models (Arhenius, Eyring, Basquin, Cox ...) to operate globally feedback data from equipments subjected to various conditions of use and environment.

It is also possible to enrich reliability databases, such as MIL-HDBK-217 or FIDES guide [4], considering only operational data, both as regards the basic failure rates as the value of parameters used in the expression of acceleration factors.

Thus, figure 5 shows the fit of the integrated circuit model of FIDES Guide, from data previously simulated.

FIDES **Integrated circuits**

	INITIAL	Adjusted
Ea_{TH} :	0,7	0,74845
Ea_{Case} :	0,12	0,1
Ea_{RH} :	0,9	0,80954
b :	0,33333	2,27878
m :	4	3,52913
n :	4,4	4,43739
o :	1,9	1,07502
t :	1,5	1,69683
p :	2,4	2,4729
$Lbd_{0\,TH}$:	60	36,0927
$Lbd_{0\,case}$:	70	100,359
$Lbd_{0\,Solder\,joints}$:	50	20,7641
$Lbd_{0\,meca}$:	30	16,1879
$Lbd_{0\,RH}$:	40	76,9813

K :	8,6E-05	
T_0 :	20	°C
F_0 :	0,08333	Hz
D_0 :	12	hr
dT_0 :	20	°C
RH_0 :	70%	
$Grms_0$:	0,5	G_{RMS}
V_0 :	100	V
$S_{reference}$:	50%	

$= \neg_{0\,TH} \exp(Ea_{TH}/k\,[1/T_0 - 1/T_j])\,(V/S_{reference}V_0)^p + \neg_{0\,case} \exp(Ea_{case}/k\,[1/T_0 - 1/T_{case}])\,(F_{cycle}/F_0)\,(dT_{cycle}/dT_0)^m$

$\neg_{0\,Solder\,joints} \exp(Ea_{case}/k\,[1/T_0 - 1/T_{Case}])\,(F_{cycle}/F_0)\,(D_{cycle}/D_0)^b\,(dT_{cycle}/dT_0)^o + \neg_{0\,Meca}\,(Grms/Grms_0)^t + \neg_{0\,RH} \exp(Ea_{RH}/k\,[1/T_0 - 1/T_{Case}])\,(RH/RH_0)^n$

	ΣLN(f(t))
	-4645,93
INITIAL :	-4649,93

T_{case}	$T_{jonction}$	F_{cycle}	D_{cycle}	dT_{cycle}	RH	Grms	V	t	AF_{TH}	Afbcase	$Af_{Solder\,joints}$	Af_{meca}	AF_{RH}	Lbd	f(t)	LN(f(t))
28,9028	51,6242	9,72945	4,66125	19,0399	0,82687	8,75254	29,2154	7110,365	4,75404	110,304	14,427071	128,66	5,3902	14039	1,3E-05	-11,2735
-17,573	1,53264	7,86378	12,025	19,7165	0,00909	29,7742	82,9139	8369,235	0,47555	50,1077	52,140812	1027,2	4E-11	22757	1,9E-05	-10,8811
75,0778	99,9576	0,0132	2,50095	35,2725	0,79623	6,88532	36,1378	10471,04	257,798	2,19511	0,0153012	85,627	282,84	32685	2,3E-05	-10,6708

Figure 5. Fit of FIDES model from simulated values

Although this model is particularly complex, it is again possible to find the approximate values of 14 different parameters used to simulate the data.

2.3. Models of maintenance

Models of preventive and corrective maintenance considering the phenomena of rejuvenation can also be adjusted in this way.

In the Generalized Renewal Process, corrective maintenance has a rejuvenating effect proportional to the time elapsed since the previous maintenance (GRP 1) or to the virtual age (GRP 2).

$$\text{GRP1: } A_r = A_{r-1} + q^*(t_r - t_{r-1}) \qquad \text{GRP2: } A_r = q^*(A_{r-1} + t_r - t_{r-1}) \tag{2}$$

Figure 6 shows the fit of GRP2 model from simulated data.

Jack [3] considered that the rejuvenation effect of equipment is more important in the case of preventive maintenance (change of several wear parts) than corrective action (change only the part down). In his model of type 2, the virtual age of equipment at the end of maintenance is equal to the one he had just before, multiplied by a proportion ρ_p in the case of a preventive maintenance or ρ_c in the case of corrective maintenance.

Figure 7 shows the fit of Jack 2 model from simulated data.

GRP 2 process (Generalized Renewal Process)

Density of probability (Weibull): $f(t) = \beta(Ar+t-t_r)^{\beta-1}/\sigma^{\beta} \exp[(A_r/\eta)^{\beta} - ([A_r+t-t_r]/\sigma)^{\beta}]$

	Initial	Ajusted
β :	2,00	1,87
σ :	500,00	493,69
q :	0,85	0,86

Adjustment **LN Likelihood:** -936,708803

Simulation $t = tr+\sigma[-\ln(ALEA())+(A_r/\sigma)^{\beta}]^{1/\beta}-A_r$

Failure (hours)	Operating time	Virtual age	LN(f(t))
532	532	459	-6,66360369
882	349	697	-6,78769657
972	90	678	-5,65854973
1051	80	653	-5,62264552
1094	43	600	-5,493481
1117	23	537	-5,47981259
1145	28	488	-5,57853328
1193	48	462	-5,69438692
1432	239	604	-6,31469755
1856	423	886	-7,41323781
1873	17	779	-5,16351285
1888	15	684	-5,25015991
2420	532	1048	-8,33797539
2448	28	928	-5,11118329
2531	83	872	-5,519183
2562	31	778	-5,24769942

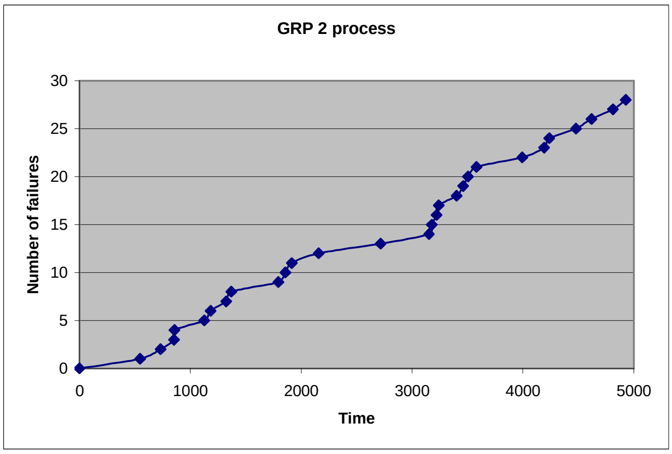

Figure 6. Fit of GRP2 model from simulated values

Aging model Jack 2

Density of probability (Weibull): $f(t) = \beta(Ar+t-t_r)^{\beta-1}/\eta^{\beta} \exp[(A_r/\eta)^{\beta} - ([A_r+t-t_r]/\eta)^{\beta}]$

Reliability: $\exp[(A_r/\eta)^{\beta} - ([A_r+t-t_r]/\eta)^{\beta}]$

	Initial	ajusted
β :	2	2,21
η :	600	623,14
ρc :	0,7	0,70
ρp :	0,3	0,26

Adjustment **LN Likelihood:** -489,7434593

Simulation $t_i = tr+\eta[-LN(ALEA())+(A_r/\eta)^{\beta}]^{1/\beta}-A_r$

Preventive (hr)	Corrective (hr)	Virtual age	Operating time	Density of probability (logarithm)	Reliability (logarithm)	LN Likelihood
	152	45	152	-7,391030033	-0,044427389	-7,391030033
300		143	148	-7,133316036	-0,071782509	-0,071782509
	303	43	3	-7,397104212	-0,001766909	-7,397104212
600		253	297	-6,633710255	-0,259924753	-0,259924753
900		411	300	-6,417985913	-0,631766755	-0,631766755
	1198	210	298	-6,417998097	-0,933283993	-6,417998097
1200		157	2	-6,951898998	-0,001630553	-0,001630553
1500		340	300	-6,473365922	-0,456695325	-0,456695325
1800		476	300	-6,408214813	-0,798549453	-0,798549453
	1945	183	145	-6,08651462	-0,439687221	-6,08651462
2100		252	155	-6,572563567	-0,192952001	-0,192952001
2400		410	300	-6,41836628	-0,629484725	-0,629484725
	2498	150	98	-6,128653362	-0,241653983	-6,128653362
2700		262	202	-6,572974171	-0,239658559	-0,239658559
3000		418	300	-6,415501097	-0,647968249	-0,647968249

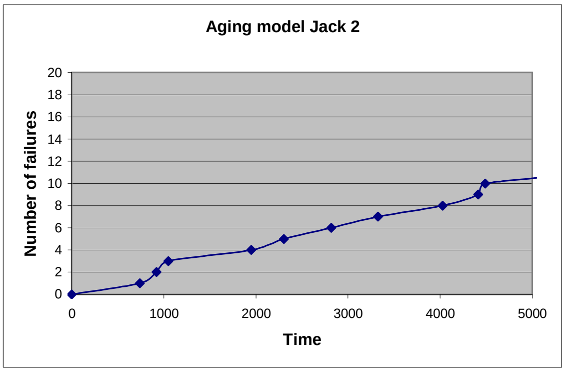

Figure 7. Fit of Jack 2 model from simulated values

2.4. Degradation model and predictive reliability model

A degradation phenomenon can be modelled by a gamma process (with increasing degradation) or a Wiener process (with possible improvements). Between two consecutive observations, the degradation rate is a random variable modelled by a gamma law in the first case and a normal law in the second one [5].

The degradation process can be stationary, if the degradation rate is independent of time, or evolutionary. In this case, the parameter α of the gamma law or the average of the normal distribution can be replaced by a parametric function of time (linear, polynomial, etc.).

An acceleration factor, such as the one of Arrhenius, can be applied to this function (or to the scale parameter of the gamma law) to account for the influence of operating conditions.

The degradation model can be transformed into predictive reliability model, by adding a threshold of acceptability of the degradation level, possibly random (e.g. Gaussian), as shown in Figure 8.

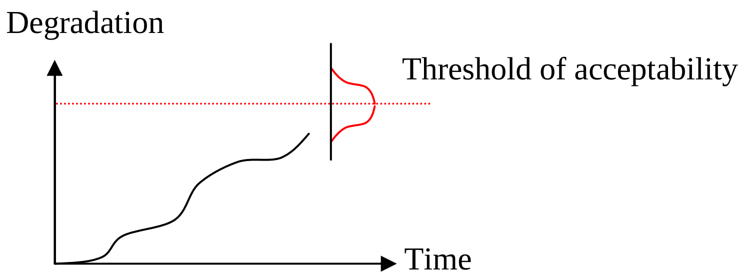

Figure 8. Degradation process

Figure 9 shows a degradation phenomenon simulated by a non-stationary gamma process performed after adjusting data feedback.

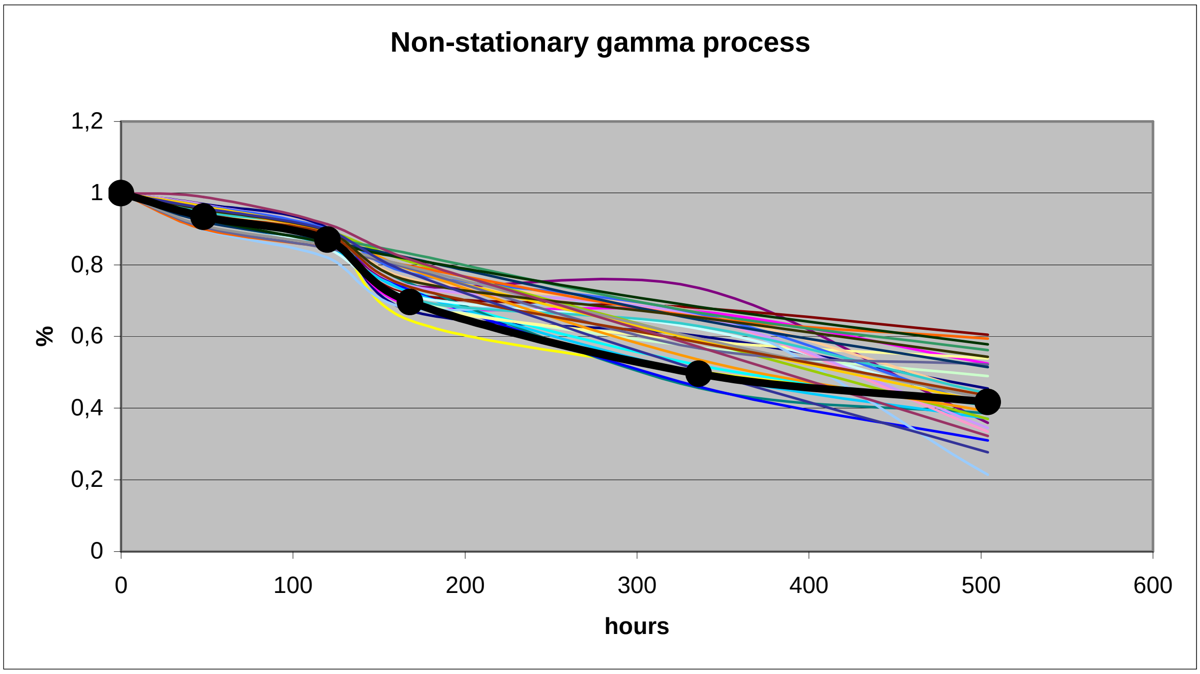

Figure 9. Simulation and data feedback of a degradation process

3. ESTIMATION OF CONFIDENCE INTERVALS IN A DISCRETE MANNER

The estimation theory allows the development of asymptotic confidence intervals from the Fisher information. These intervals can be used only above a minimum sample size for which thirty elements is a commonly accepted value. Indeed, the number of elements has to tend to infinity so that we reach the required level of confidence without control of the speed of convergence.

3.1. Asymptotic confidence intervals

The Fisher information is the variance of the score function, which is the gradient (partial derivative) of the logarithm of the likelihood function.

$$I_n(\theta) = \mathrm{E}\left(\left[\frac{\partial Ln\, L(X,\theta)}{\partial\theta}\right]^2\right) = -\mathrm{E}\left(\frac{\partial^2 Ln\, L(X,\theta)}{\partial\theta^2}\right) \qquad (3)$$

As a first approximation, the maximum likelihood estimator of θ is distributed according to a Gaussian whose variance is equal to the inverse of the Fisher information.

The asymptotic confidence interval of Θ, with confidence level equal to β = 1-α, is then:

$$[\hat{\theta}_n - \frac{1}{\sqrt{n\, I(\theta)}} z_{1-\alpha/2} ; \hat{\theta}_n + \frac{1}{\sqrt{n\, I(\theta)}} z_{1-\alpha/2}] \qquad (4)$$

with $Z_{1-\alpha/2}$ the distribution function of standard normal distribution function.

In the case of a probabilistic model with more parameters, the Fisher information matrix is defined as follows:

$$F_{ij} = I_n(\theta)_{ij} = \mathrm{E}\left(\left[\frac{\partial Ln\, L(X,\theta)}{\partial\theta_i}\right]\left[\frac{\partial Ln\, L(X,\theta)}{\partial\theta_j}\right]\right) = -\mathrm{E}\left(\frac{\partial^2 Ln\, L(X,\theta)}{\partial\theta_i \partial\theta_j}\right) \qquad (5)$$

Or in the case of three parameters:

$$F = I_n(\theta) = \begin{pmatrix} -\frac{\partial^2 Ln\, L(X,\theta)}{\partial\theta_1{}^2} & -\frac{\partial^2 Ln\, L(X,\theta)}{\partial\theta_1\, \partial\theta_2} & -\frac{\partial^2 Ln\, L(X,\theta)}{\partial\theta_1\, \partial\theta_3} \\ -\frac{\partial^2 Ln\, L(X,\theta)}{\partial\theta_2\, \partial\theta_1} & -\frac{\partial^2 Ln\, L(X,\theta)}{\partial\theta_2{}^2} & -\frac{\partial^2 Ln\, L(X,\theta)}{\partial\theta_2\, \partial\theta_3} \\ -\frac{\partial^2 Ln\, L(X,\theta)}{\partial\theta_3\, \partial\theta_1} & -\frac{\partial^2 Ln\, L(X,\theta)}{\partial\theta_3\, \partial\theta_2} & -\frac{\partial^2 Ln\, L(X,\theta)}{\partial\theta_3{}^2} \end{pmatrix} \qquad (6)$$

The inverse of the Fisher information matrix is an approximation of the covariance matrix of maximum likelihood estimators.

An asymptotic confidence interval can then be calculated for each parameter from the diagonal elements of this matrix (variance).

Similarly, an asymptotic confidence interval can be calculated for any function g(θ) of the different parameters (quantile, distribution function ...) by calculating the variance of this function as follows:

$$\sigma^2_{\hat{g}_n}(\theta) = \nabla g(\theta)^T\ I_n^{-1}(\theta)\, \nabla g(\theta) \qquad (7)$$

with $\nabla g(\theta)\ et\, \nabla g(\theta)^T$ the gradient of g and its transpose and $I_n^{-1}(\theta)$ the inverse of the Fisher matrix

3.2. Discrete calculation method

To avoid the calculation of second derivatives and partial derivatives in the Fischer matrix, it is possible to make these calculations in a generic manner for any probability models by using a discrete calculation method based on Taylor formulas.

The second derivatives in Θ_i of the log likelihood function can be calculated as follows from a small increment h:

$$\begin{aligned}
&f(\hat{\theta}_i + h) = f(\hat{\theta}_i) + \frac{\partial f}{\partial \theta_i}(\hat{\theta}_i) \times h + \frac{\partial^2 f}{\partial {\theta_i}^2}(\hat{\theta}_i) \times \frac{h^2}{2} + \varepsilon(h) \\
&f(\hat{\theta}_i - h) = f(\hat{\theta}_i) - \frac{\partial f}{\partial \theta_i}(\hat{\theta}_i) \times h + \frac{\partial^2 f}{\partial {\theta_i}^2}(\hat{\theta}_i) \times \frac{h^2}{2} + \varepsilon(h) \\
&f(\hat{\theta}_i + h) + f(\hat{\theta}_i - h) = 2 f(\hat{\theta}_i) + \frac{\partial^2 f}{\partial {\theta_i}^2}(\hat{\theta}_i) \times h^2 + 2\varepsilon(h) \\
&\Rightarrow \quad \frac{\partial^2 f}{\partial {\theta_i}^2}(\hat{\theta}_i) = \lim_{h \to 0} \frac{f(\hat{\theta}_i + h) + f(\hat{\theta}_i - h) - 2 f(\hat{\theta}_i)}{h^2}
\end{aligned} \tag{8}$$

Similarly, the partial derivatives in Θ_i Θ_j can be calculated as follows using small increments h and k:

$$f(\hat{\theta}_i + h, \hat{\theta}_j + k) = f(\hat{\theta}_i, \hat{\theta}_j) + h \times \frac{\partial f}{\partial \theta_i}(\hat{\theta}_i, \hat{\theta}_j) + k \times \frac{\partial f}{\partial \theta_j}(\hat{\theta}_i, \hat{\theta}_j) + \frac{1}{2}[h^2 \times \frac{\partial^2 f}{\partial {\theta_i}^2}(\hat{\theta}_i, \hat{\theta}_j) + 2hk \times \frac{\partial^2 f}{\partial \theta_i \partial \theta_j}(\hat{\theta}_i, \hat{\theta}_j) + k^2 \times \frac{\partial^2 f}{\partial {\theta_j}^2}(\hat{\theta}_i, \hat{\theta}_j)] + (h^2 + k^2)\, \varepsilon(h,k)$$

$$f(\hat{\theta}_i - h, \hat{\theta}_j - k) = f(\hat{\theta}_i, \hat{\theta}_j) - h \times \frac{\partial f}{\partial \theta_i}(\hat{\theta}_i, \hat{\theta}_j) - k \times \frac{\partial f}{\partial \theta_j}(\hat{\theta}_i, \hat{\theta}_j) + \frac{1}{2}[h^2 \times \frac{\partial^2 f}{\partial {\theta_i}^2}(\hat{\theta}_i, \hat{\theta}_j) + 2hk \times \frac{\partial^2 f}{\partial \theta_i \partial \theta_j}(\hat{\theta}_i, \hat{\theta}_j) + k^2 \times \frac{\partial^2 f}{\partial {\theta_j}^2}(\hat{\theta}_i, \hat{\theta}_j)] + (h^2 + k^2)\, \varepsilon(h,k)$$

$$f(\hat{\theta}_i + h, \hat{\theta}_j + k) + f(\hat{\theta}_i - h, \hat{\theta}_j - k) = 2 f(\hat{\theta}_i, \hat{\theta}_j) + h^2 \times \frac{\partial^2 f}{\partial {\theta_i}^2}(\hat{\theta}_i, \hat{\theta}_j) + 2hk \times \frac{\partial^2 f}{\partial \theta_i \partial \theta_j}(\hat{\theta}_i, \hat{\theta}_j) + k^2 \times \frac{\partial^2 f}{\partial {\theta_j}^2}(\hat{\theta}_i, \hat{\theta}_j) + 2(h^2 + k^2)\, \varepsilon(h,k)$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial \theta_i \partial \theta_j}(\hat{\theta}_i, \hat{\theta}_j) = \lim_{(h,k) \to (0,0)} \frac{f(\hat{\theta}_i + h, \hat{\theta}_j + k) + f(\hat{\theta}_i - h, \hat{\theta}_j - k) - 2 f(\hat{\theta}_i, \hat{\theta}_j) - h^2 \times \frac{\partial^2 f}{\partial {\theta_i}^2}(\hat{\theta}_i, \hat{\theta}_j) - k^2 \times \frac{\partial^2 f}{\partial {\theta_j}^2}(\hat{\theta}_i, \hat{\theta}_j)}{2hk} \tag{9}$$

This calculation method was implemented in the GENCAB software and its accuracy was verified by comparing its results with those obtained by the Fisher matrices computed "by hand" by deriving analytical expressions as follows in the case of Weibull law:

$$L(X; \beta, \sigma, \gamma) = \prod_{i=1}^{n} f(X_i; \beta, \sigma, \gamma) \;\; with \;\; f(X_i; \beta, \sigma, \gamma) = \frac{\beta}{\sigma}(\frac{X_i - \gamma}{\sigma})^{\beta - 1} \exp(-(\frac{X_i - \gamma}{\sigma})^{\beta}) \tag{10}$$

$$I_n(\beta, \sigma, \gamma) = \begin{pmatrix}
\frac{n}{\beta^2} + \sum_{i=1}^{n} (\frac{x_i - \gamma}{\sigma})^{\beta} [Ln(\frac{x_i - \gamma}{\sigma})]^2 & \frac{n}{\sigma} - \sum_{i=1}^{n} \left[\frac{(X_i - \gamma)^{\beta}}{\sigma^{\beta+1}} \times (1 + \beta \times Ln(\frac{X_i - \gamma}{\sigma})) \right] & -\sum_{i=1}^{n} \frac{(X_i - \gamma)^{\beta-1}}{\sigma^{\beta}} \left[1 + \beta \times Ln(\frac{X_i - \gamma}{\sigma}) \right] - \frac{1}{(X_i - \gamma)} \\
\frac{n}{\sigma} - \sum_{i=1}^{n} \left[\frac{(X_i - \gamma)^{\beta}}{\sigma^{\beta+1}} \times (1 + \beta \times Ln(\frac{X_i - \gamma}{\sigma})) \right] & \frac{-n\beta}{\sigma^2} + \frac{\beta(\beta+1) \sum_{i=1}^{n} (x_i - \gamma)^{\beta}}{\sigma^{\beta+2}} & \frac{\beta^2}{\sigma^{\beta+1}} \sum_{i=1}^{n} (X_i - \gamma)^{\beta-1} \\
-\sum_{i=1}^{n} \frac{(X_i - \gamma)^{\beta-1}}{\sigma^{\beta}} \left[1 + \beta \times Ln(\frac{X_i - \gamma}{\sigma}) \right] - \frac{1}{(X_i - \gamma)} & \frac{\beta^2}{\sigma^{\beta+1}} \sum_{i=1}^{n} (X_i - \gamma)^{\beta-1} & -(1-\beta) \sum_{i=1}^{n} \frac{1}{(X_i - \gamma)^2} - \frac{\beta(\beta-1)}{\sigma^{\beta}} \sum_{i=1}^{n} (X_i - \gamma)^{\beta-2}
\end{pmatrix} \tag{11}$$

The results obtained are very close as shown in Figure 10.

Adjustment *Maximum Likelihood*

Probability law: WEIBULL (3 parameters)

Bêta : 2,7038961
Sigma : 7,6776398
Gamma : -1,522354

LN Likelihood -121,37057

Uncensored

LN K (uncensored) -121,37057

Variable	Rate : λ(ti)	R(ti) = 1-F(ti)	Density : f(ti)	Ln(f(ti))
6,5259554	0,3816391	0,3211099	0,1225481	-2,0992517
9,5245605	0,6546369	0,0689375	0,045129	-3,0982291
5,7520005	0,3212436	0,4213679	0,1353617	-1,9998047
9,6390822	0,6662425	0,0639158	0,0425834	-3,15629
3,7060104	0,1829983	0,7019782	0,1284608	-2,0521316
1,0922939	0,0561895	0,9471149	0,0532179	-2,9333606
1,7002497	0,0802339	0,9088041	0,0729169	-2,6184347
6,9386172	0,4155793	0,272417	0,1132108	-2,1785033
6,3421936	0,3669114	0,343971	0,1262069	-2,0698328
8,0510675	0,5129355	0,1626588	0,0834334	-2,483706
8,039598	0,5118888	0,1636175	0,083754	-2,4798714
9,2845503	0,6305882	0,0804332	0,0507202	-2,9814307
4,6043293	0,2397599	0,5808485	0,1392642	-1,9713826
6,5910941	0,386917	0,3131719	0,1211716	-2,110548

Fisher Matrix by GENCAB

11,054695	-2,54619	2,457989
-2,546189	6,201497	5,573605
2,4579892	5,573605	7,393015

determinant 8,2560084

Fisher Matrix "by hand"

11,054649	-2,54611	2,457994
-2,546111	6,201468	5,573579
2,4579943	5,573579	7,393013

determinant 8,26142062

$$I_n(\beta,\sigma,\gamma)=\begin{pmatrix} \frac{n}{\beta^2}+\sum_{i=1}^{n}(\frac{x_i-\gamma}{\sigma})^{\beta}[Ln(\frac{x_i-\gamma}{\sigma})]^2 & \frac{n}{\sigma}-\sum_{i=1}^{n}\left[\frac{(X_i-\gamma)^{\beta}}{\sigma^{\beta+1}}\times(1+\beta\times Ln(\frac{X_i-\gamma}{\sigma}))\right] & -\sum_{i=1}^{n}\frac{(X_i-\gamma)^{\beta-1}}{\sigma^{\beta}}\left[1+\beta\times Ln(\frac{X_i-\gamma}{\sigma})\right]-\frac{1}{(X_i-\gamma)} \\ \frac{n}{\sigma}-\sum_{i=1}^{n}\left[\frac{(X_i-\gamma)^{\beta}}{\sigma^{\beta+1}}\times(1+\beta\times Ln(\frac{X_i-\gamma}{\sigma}))\right] & \frac{-n\beta}{\sigma^2}+\frac{\beta(\beta+1)\sum_{i=1}^{n}(x_i-\gamma)^{\beta}}{\sigma^{\beta+2}} & \frac{\beta^2}{\sigma^{\beta+1}}\sum_{i=1}^{n}(X_i-\gamma)^{\beta-1} \\ -\sum_{i=1}^{n}\frac{(X_i-\gamma)^{\beta-1}}{\sigma^{\beta}}\left[1+\beta\times Ln(\frac{X_i-\gamma}{\sigma})\right]-\frac{1}{(X_i-\gamma)} & \frac{\beta^2}{\sigma^{\beta+1}}\sum_{i=1}^{n}(X_i-\gamma)^{\beta-1} & -(1-\beta)\sum_{i=1}^{n}\frac{1}{(X_i-\gamma)^2}-\frac{\beta(\beta-1)}{\sigma^{\beta}}\sum_{i=1}^{n}(X_i-\gamma)^{\beta-2} \end{pmatrix}$$

Figure 10. Comparison of results for the Weibull law

4. CONCLUSION

A global optimization tool allows complex probabilistic models to be adjusted and thus facilitates the use of data feedback.

It is particularly valuable for predictive maintenance and health monitoring which are a real “Graal” for the reliability-community. Indeed, this makes it possible to space out the maintenance actions (on average) while paradoxically reducing the risk of failure.

In addition, asymptotic confidence intervals can be calculated by a generic method, based on the Taylor formula, which gives accurate results.

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